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The Summation of Series Whose Terms Have Asymptotic Representations

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I. INTRODUCTION

Previous writers have discussed the expansion of the Meijer G-function

$$f(x) = G_{p,q}^{m,k}\left(x \,\middle|\, {}_{b_q}^{a_p}\right) \tag{1.1}$$

in series of the classical orthogonal polynomials of argument λ/x; see [1, 2]. (For the theory of the special functions used in this paper, the Erdélyi volumes [3] provide a good reference.) Without giving conditions, we write the pertinent expansions in the form

$$f(x) = \sum_{n=0}^{\infty} C_n(\lambda)\, R_n^{(\alpha,\beta)}\left(\frac{\lambda}{x}\right) \qquad 1 \leqslant x/\lambda \leqslant \infty \tag{1.2}$$

and

$$f(x) = \sum_{n=0}^{\infty} D_n(\lambda)\, L_n^{(\alpha)}\left(\frac{\lambda}{x}\right) \qquad 0 < \lambda/x < \infty. \tag{1.3}$$

We have used the notation $R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x - 1)$ for the shifted Jacobi polynomial. It has also been shown [4, 5] that for particularly important values of m, k, p, and q the coefficients C_n, D_n may be represented asymptotically by

$$\left.\begin{aligned} C_n(\lambda) &= d_1 n^{\theta_1} e^{-a_1 n^{2/\rho}}\,[1 + O(n^{-\delta_1})], \qquad n \to \infty \\ D_n(\lambda) &= d_2 n^{\theta_2} e^{-a_2 n^{2/\rho}}\,[1 + O(n^{-\delta_2})], \qquad n \to \infty \end{aligned}\right\} \tag{1.4}$$

where

$$\rho = q - p + 2 \geqslant 3.$$

Furthermore, in [6, 7], C_n, D_n are shown to satisfy certain difference equations whose coefficients are polynomials in n. In fact, these difference equations are of the so-called irregular type, first studied by writers such as Adams [8] and Batchelder [9]. The theory of these equations was completed by Birkhoff and Trjitzinsky in two of the most important and most overlooked papers of classical analysis, see [10] and [11]. Briefly, these authors showed that any function which satisfies a difference equation in ω whose coefficients possess convergent or even asymptotic expansions in powers of $\omega^{-1/r}$, r an integer, will possess an asymptotic expansion given by a linear combination of series of the form

$$y(\omega) \sim e^{Q(\omega)} \sum_{j=0}^{m} (\ln \omega)^j s_j(\omega), \qquad \omega \to \infty, \tag{1.5}$$

where

$$Q(\omega) = \mu_0 \omega \ln \omega + \mu_1 \omega + \mu_2 \omega^{(\rho-1)/\rho} + \mu_3 \omega^{(\rho-2)/\rho} + \cdots + \mu_\rho \omega^{1/\rho}, \tag{1.6}$$

$$s_j(\omega) = \omega^\theta[\alpha_{0,j} + \alpha_{1,j}\omega^{-1/\rho} + \alpha_{2,j}\omega^{-2/\rho} + \cdots], \tag{1.7}$$

and ρ is an integral multiple of r, μ_0 an integral multiple of $1/\rho$. In this context, the term "asymptotic expansion" has the following meaning. The functions $\{(\ln \omega)^s \omega^{-t/\rho}\}$ $t = 0, 1, 2, \ldots$, $s = 0, 1, 2, \ldots, m$, can be arranged in descending order of growth as $\omega \to \infty$. Let this arrangement be $\{x_0(\omega), x_1(\omega), x_2(\omega), x_3(\omega) \cdots\}$. The above expansion can be written

$$e^{Q(\omega)}\omega^\theta \sum_{j=0}^{\infty} x_j(\omega), \tag{1.8}$$

and what we mean by (1.5) is

$$e^{-Q(\omega)}\omega^{-\theta} y(\omega) - \sum_{j=0}^{r} x_j(\omega) = 0[x_{r+1}(\omega)], \; \omega \to \infty. \tag{1.9}$$

It is easy to show that such Birkhoff expansions, as we shall call them, are unique, like the ordinary Poincaré asymptotic series ($Q \equiv 0, \rho = 1$). The actual coefficients of the expansion are determined by substitution using such elementary identities as

$$(\omega + k)^\alpha = \omega^\alpha[1 + (\alpha k/\omega) + \cdots] \tag{1.10}$$

$$\begin{aligned}[\ln(\omega + k)]^s &= \left[\ln \omega + \ln\left(1 + \frac{k}{\omega}\right)\right]^s \\ &= (\ln \omega)^s + s(\ln \omega)^{s-1}\left(\frac{k}{\omega} - \frac{k^2}{2\omega^2} + \cdots\right) + \cdots \end{aligned} \tag{1.11}$$

and forcing the coefficients of $x_j(\omega)$ to vanish, see Birkhoff [10].

In view of these observations, we can see that the isolated facts about the Jacobi polynomial expansions begin to fit together. For instance, the steepest descent analyses in the references [4, 5] which yielded the equations (1.4) provided, in fact, the leading terms of the Birkhoff expansions for $C_n(\lambda)$ and $D_n(\lambda)$. Once these leading terms (or connecting constants for the expansions) are known, then the higher order terms can be obtained from the difference equation by the purely algebraic methods discussed above.

Now, an excellent way of computing $f(x)$ is to let $x = \lambda$ in (1.2) (the expansion (1.3) is in general much less rapidly convergent). Since

$$R_n^{(\alpha,\beta)}(1) = (\alpha + 1)_n/n! \tag{1.12}$$

we will then have a series which is rapidly convergent, which does not involve Jacobi polynomials, and whose terms possess Birkhoff expansions in n. Of course, many other series of this type are often encountered in practice.

Methods for summing such series that would take advantage of the fact that the n-th term possessed an asymptotic expansion of the Birkhoff type would be of great general interest; and in the first part of this paper we discuss such methods.

We shall see that these methods can be used to derive asymptotic expansions not only for the remainder of the series $\sum a_n$ where a_n has a Birkhoff representation, but also for the remainder of series of the form $\sum a_n P_n$, where a_n has a Birkhoff representation and P_n is a classical orthogonal polynomial. The reason for this is that P_n must satisfy a recursion relation of order three with polynomial coefficients, and hence itself may be represented as the sum of two Birkhoff series.

II. Applications to Ordinary Series

We first require the following.

THEOREM 1 (Birkhoff-Trjitzinsky). *Let the series*

$$A = \sum_{n=0}^{\infty} a_n \tag{2.1}$$

converge. Let

$$S_n = \sum_{r=0}^{n-1} a_r, \qquad R_n = A - S_n \tag{2.2}$$

and let

$$a_n = a(\omega) \sim e^{Q(\omega)} s(\omega), \qquad \omega = n + \zeta, \quad n \to \infty, \tag{2.3}$$

$Q(\omega)$ as above and

$$s(\omega) = \omega^{\theta}(\alpha_0 + \alpha_1\omega^{-1/\rho} + \alpha_2\omega^{-2/\rho} + \cdots). \tag{2.4}$$

Then R_n possesses the asymptotic representation

$$R_n = R(\omega) \sim e^{Q(\omega)}s^*(\omega), \qquad n \to \infty \tag{2.5}$$

where

$$s^*(\omega) = \omega^{\theta^*} \sum_{m=0}^{\infty} \beta_m \omega^{-m/\rho}. \tag{2.6}$$

Furthermore, θ^*, β_0 are as follows.

Case I: $Q \not\equiv 0$. Let the first nonzero μ_j in the sequence $[\mu_0, \mu_1, ..., \mu_\rho]$ be denoted by μ_τ. Then

$$\theta^* = \begin{cases} \theta, & \tau = 0 \\ \theta + (\tau - 1)/\rho, & 1 \leqslant \tau \leqslant \rho \end{cases} \tag{2.7}$$

$$\beta_0 = \begin{cases} -\alpha_0, & \tau = 0 \\ \alpha_0/(1 - e^{\mu_1}), & \tau = 1 \\ -\alpha_0\rho/[\mu_\tau(\rho + 1 - \tau)], & 2 \leqslant \tau \leqslant \rho \end{cases} \tag{2.8}$$

Case II: $Q \equiv 0$.

$$\theta^* = \theta + 1, \qquad \beta_0 = \alpha_0/(\theta + 1). \tag{2.9}$$

Proof. We have

$$R_{n+1} - R_n = -a_n \tag{2.10}$$

or

$$R(\omega + 1) - R(\omega) = -a(\omega), \qquad \omega = n + \alpha, \tag{2.11}$$

and the conclusion (2.5) follows directly from Lemma 8 in [11, p. 30]. There are, however, two points that require clarification. First, there is an arbitrary constant to be added to the series for $R(\omega)$, but the constant must be zero since $R(\omega) \to 0$. Next, no logarithms can occur in the series for $R(\omega)$, since logarithms will occur if and only if $Q \equiv 0$ and $s(\omega)$ contains a term ω^{-1}, see Birkhoff [10, p. 220]. However, such a case is excluded by the requirement that $\sum a_n$ converge. The actual determination of the constants θ^*, β_m in $s^*(\omega)$ is accomplished by using simple identities such as (1.10, 1.11), see Birkhoff [10] for examples.

We find, for instance, that

$$e^{\Delta Q(\omega)} = \omega^{\mu_0} e^{\mu_0 + \mu_1}[1 + ((\rho + 1 - \sigma)/\rho)\, \mu_\sigma \omega^{(1-\sigma)/\rho} + O(\omega^{-\sigma/\rho})] \quad (2.12)$$
$$Q \not\equiv 0,$$

where μ_σ is the first nonzero μ_j in the sequence $\mu_2, \mu_3, \ldots, \mu_\rho$.

From the difference equation (2.11) it follows that we must have

$$\{\omega^{\mu_0} e^{\mu_0+\mu_1}[1 + ((\rho + 1 - \sigma/\rho)\, \mu_\sigma \omega^{(1-\sigma)/\rho} + O(\omega^{-\sigma/\rho})] - 1\}$$
$$\times \{\beta_0 + \beta_1 \omega^{-1/\rho} + \cdots + \beta_\rho \omega^{-1} + O(\omega^{-1-1/\rho})\}$$
$$= -\omega^{\theta-\theta^*}\{\alpha_0 + O(\omega^{-1/\rho})\}. \quad (2.13)$$

The requirement that the leading coefficient of both sides agree, leads to (2.7), (2.8). (Note that $\mu_0 \leqslant 0$).

If $Q \equiv 0$, the computations are even easier and we find

$$(\theta^*\beta_0/\omega) + (-(\beta_1/\rho) + \theta^*\beta_1)\, \omega^{-1-1/\rho} + \cdots = \omega^{\theta-\theta^*}(\alpha_0 + \alpha_1\omega^{-1/\rho} + \cdots) \quad (2.14)$$

from which follows (2.9).

THEOREM 2. *Let* (2.1), (2.2) *hold but let*

$$a(\omega) \sim e^{Q(\omega)}(\ln \omega)^p \sum_{m=0}^{\infty} \beta_m \omega^{-m/\rho} \quad (2.15)$$

p a positive integer. Then the asymptotic expansion of $R(\omega)$ may be obtained by formally differentiating the series $e^{Q(\omega)}s^*(\omega)$ p times with respect to θ. Similar asymptotic estimates for $R(\omega)$ may be obtained when $a(\omega)$ is any linear combination of series of the kind (2.15) *for different values of p.*

Proof. The above follows from the same Lemma in [11, p. 30].

To analyze the error, we introduce the following quantities

$$s_k^*(\omega) = \omega^{\theta^*} \sum_{m=0}^{k-1} \beta_m \omega^{-m/\rho}, \qquad k = 1, 2, 3, \ldots, \qquad \operatorname{Re} \omega > 0, \quad (2.16)$$

β_m, θ^* as above,

$$R(\omega) = e^{Q(\omega)}[s_k^*(\omega) + \epsilon_k(\omega)\omega^{-k/\rho}], \qquad \operatorname{Re} \omega > 0 \quad (2.17)$$

so $\epsilon_k(\omega)$ is a measure of the error of the process. Further let

$$e^{Q(\omega)}\eta(\omega) = -a(\omega) - \Delta[e^{Q(\omega)}s_k^*(\omega)], \qquad \operatorname{Re} \omega > 0. \quad (2.18)$$

Thus, once $s_k{}^*(\omega)$ has been calculated, $\eta(\omega)$ is known. As $n \to \infty$ we have

$$e^{Q(\omega)}\eta(\omega) \sim \Delta[e^{Q(\omega)}(s^*(\omega) - s_k{}^*(\omega))] = e^{Q(\omega)}O(\omega^{\theta^*-k/\rho}) \qquad (2.19)$$

at least.

Hence for Re $(\omega) \geqslant N > 0$ we can determine η_N such that

$$|\,\eta(\omega)| \leqslant \eta_N \,|\, \omega\,|^{\mathrm{Re}\,\theta^*-k/\rho}, \qquad \mathrm{Re}\,(\omega) \geqslant N > 0. \qquad (2.20)$$

It is easily verified that $\epsilon_k(\omega)$ satisfies

$$q(\omega+1)\,\epsilon_k(\omega+1) - q(\omega)\,\epsilon_k(\omega) = e^{Q(\omega)}\eta(\omega), \qquad \mathrm{Re}\,\omega > 0, \quad (2.21)$$

where

$$q(\omega) = e^{Q(\omega)}\omega^{-k/\rho}, \qquad \mathrm{Re}\,\omega > 0. \qquad (2.22)$$

Thus,

$$\epsilon_k(\omega) = -\omega^{k/\rho}\, e^{-Q(\omega)} \left\{ \sum_{s=0}^{\infty} e^{Q(\omega+s)}\, \eta(\omega+s) + M \right\}, \qquad \mathrm{Re}\,\omega > 0, \quad (2.23)$$

as may be verified by differencing. Furthermore, M must be zero, since $\epsilon_k(\omega) \to 0$ as $n \to \infty$ by Theorem 1. Then

$$|\,\epsilon_k(\omega)| \leqslant |\,\omega\,|^{k/\rho}\, e^{-\mathrm{Re}\;Q(\omega)}\, \eta_N \sum_{s=0}^{\infty} e^{\mathrm{Re}\;Q(\omega+s)}\, |\,\omega+s\,|^{\mathrm{Re}\;Q^*-k/\rho}, \quad \mathrm{Re}\,\omega \geqslant N. \qquad (2.24)$$

We distinguish two cases. First, let Re $Q = 0$. Then $\theta^* = \theta + 1$ and

$$|\,\epsilon_k(\omega)| \leqslant |\,\omega\,|^{k/\rho}\, \eta_N \sum_{s=0}^{\infty} |\,\omega+s\,|^{\mathrm{Re}\;\theta+1-k/\rho}. \qquad (2.25)$$

Next, let Re $Q \not\equiv 0$. Then there exists an $N' \geqslant N$ such that

$$e^{\mathrm{Re}\,Q(\omega)}\,|\,\omega\,|^{\mathrm{Re}\,\theta-(k/\rho)+2} \qquad (2.26)$$

is monotone decreasing for $\omega \geqslant N'$. Thus

$$|\,\epsilon_k(\omega)| \leqslant \eta_N\,|\,\omega\,|^{\mathrm{Re}\;\theta^*+2} \sum_{s=0}^{\infty} |\,\omega+s\,|^{-2}, \quad \omega > N'. \qquad (2.27)$$

Our next theorem follows from the expressions (2.25) and (2.27) by an application of the inequality [11, p. 33].

$$\sum_{s=0}^{\infty} \frac{1}{|\,x+s\,|^{k'}} < \frac{\pi}{2\,|\,x-1\,|^{k'-1}}, \quad \mathrm{Re}\,x > 1, \quad k' > 1. \qquad (2.28)$$

THEOREM 3. *Let η_N, $\eta(\omega)$, $\epsilon_k(\omega)$, etc. be as above.*

(A) *Let* $\operatorname{Re} Q(\omega) \equiv 0$, $k \geqslant \rho$, $\operatorname{Re} \omega \geqslant N > 1$. *Then*

$$| \epsilon_k(\omega)| \leqslant (\pi/2)\, \eta_N \,| \omega |^{k/\rho} \,| \omega - 1 |^{\operatorname{Re} \theta+2-k/\rho}. \tag{2.29}$$

(B) *Let* $\operatorname{Re} Q(\omega) \not\equiv 0$, *$N'$ be as above. Then*

$$| \epsilon_k(\omega)| \leqslant (\pi/2)\, \eta_N \,| \omega |^{\operatorname{Re} \theta^*+2} \,| \omega - 1 |^{-1}, \operatorname{Re} \omega \geqslant N' \geqslant N > 1, \tag{2.30}$$

where θ^* is given by (2.7).

Remark. In either case, we can say that $\epsilon_k(\omega) = O[\omega^{\theta^*+1}]$ as $n \to \infty$. Also, of interest is the case where $\sum a_n$ diverges. The following theorem gives an asymptotic expansion for the nth partial sum of the series.

THEOREM 4. *Let $\sum a_n$ diverge. Let*

$$S_n = S(\omega) \tag{2.31}$$

and a_n, $a(\omega)$, $s^*(\omega)$ be as in Theorem 1. *Then*

$$S(\omega) \sim -e^{Q(\omega)} s^*(\omega) + C, \qquad n \to \infty, \tag{2.32}$$

unless $Q \equiv 0$ and $S(\omega)$ contains a term ω^{-1}. In this case,

$$S(\omega) \sim M \ln \omega + \hat{s}(\omega) + C, \qquad n \to \infty, \tag{2.33}$$

where $\hat{s}(\omega)$ is a series of the kind (2.6) *with $\theta^* = \theta + 1$.*

Proof. Use the equation

$$\Delta S(\omega) = a(\omega), \tag{2.34}$$

and proceed as before. The statement for the case $Q \equiv 0$ follows from an observation of Birkhoff [10, p. 220].

When $Q = \mu_1 \omega$, the coefficients β_m in Theorem 1 can be found in closed form as follows. In forming $\Delta R(\omega)$ we encounter the sum

$$\sum_{m=0}^{\infty} \beta_m \omega^{-m/\rho} \left(1 + \frac{1}{\omega}\right)^{\theta^*-m/\rho}$$

$$= \sum_{m=0}^{\infty} \beta_m \omega^{-m/\rho} \sum_{r=0}^{\infty} \frac{\left(\frac{m}{\rho} - \theta^*\right)_{r/\rho} (-1)^{r/\rho} e_r \omega^{-r/\rho}}{\Gamma((r/\rho) + 1)} \tag{2.35}$$

where

$$e_r = \begin{cases} 1, & \rho \mid r \\ 0, & \text{otherwise.} \end{cases} \tag{2.36}$$

It is found that the β_j's must satisfy the equation

$$e^{\mu_1} \sum_{m=0}^{r} \frac{\Gamma\left(\frac{r}{\rho} - \theta^*\right)(-1)^{(r-m)/\rho}\, e_{r-m}\beta_m}{\Gamma\left(\frac{m}{\rho} - \theta^*\right)\Gamma\left(\frac{r-m}{\rho} + 1\right)} - \beta_r = \begin{cases} -\alpha_r, & \mu_1 \neq 0 \\ -\alpha_{r-\rho}, & \mu_1 = 0 \end{cases} \tag{2.37}$$

This equation may be solved by generating functions. First let

$$\begin{aligned} Z(\omega) &= \sum_{r=0}^{\infty} \omega^{r/\rho}\zeta_r, \qquad A(\omega) = \sum_{r=0}^{\infty} \omega^{r/\rho} A_r, \\ \zeta_r &= \beta_r/\Gamma((r/\rho) - \theta), \qquad A_r = -\alpha_r/\Gamma((r/\rho) - \theta). \end{aligned} \tag{2.38}$$

Multiplying both sides by $\omega^{r/\rho}$ and summing from $r = 0$ to $r = \infty$ gives the formal relationship

$$e^{\mu_1} Z(\omega) \sum_{r=0}^{\infty} \frac{\omega^{r/\rho}(-1)^{r/\rho}\, e_r}{\Gamma((r/\rho) + 1)} - Z(\omega) = A(\omega) \tag{2.39}$$

or

$$\begin{aligned} Z(\omega) &= A(\omega)(e^{\mu_1 - \omega} - 1)^{-1} = A(\omega) \sum_{s=0}^{\infty} \frac{e^{\omega}(e^{\omega} - 1)^s}{(e^{\mu_1} - 1)^{s+1}} \\ &= A(\omega) \sum_{\nu=0}^{\infty} \omega^{\nu/\rho} e_\nu \sum_{s=0}^{\nu/\rho} \frac{B^{(-s)}_{(\nu/\rho)-s}(1)}{((\nu/\rho) - s)!\,(e^{\mu_1} - 1)^{s+1}} \end{aligned} \tag{2.40}$$

By [12, p. 145] we have

$$B^{(-s)}_{(\nu/\rho)-s}(1) = \frac{(1 + \nu/\rho)\, B_{(\nu/\rho)-s}}{(s + 1)}, \qquad \rho \mid \nu, \tag{2.41}$$

so selecting the coefficient of $\omega^{r/\rho}$ on the right-hand side gives

$$\begin{aligned} \beta_r &= -\sum_{\nu=0}^{r} \alpha_{r-\nu} \left(\frac{r-\nu}{\rho} - \theta\right)_{\nu/\rho} e_\nu (1 + \nu/\rho) \sum_{s=0}^{\nu/\rho} \\ &\quad \times \frac{B^{(-s-1)}_{(\nu/\rho)-s}}{(s + 1)((\nu/\rho) - s)!\,(e^{\mu_1} - 1)^{s+1}} \\ &= -\sum_{m=0}^{\leqslant r/\rho} \alpha_{r-m\rho} \left(\frac{r}{\rho} - m - \theta\right)_m (m + 1) \sum_{s=0}^{m} \frac{B^{(-s-1)}_{m-s}}{(s + 1)\,(m - s)!\,(e^{\mu_1} - 1)^{s+1}} \end{aligned} \tag{2.42}$$

where $(\alpha)_\sigma$ is Pochhammer's symbol,

$$(\alpha)_\sigma = \Gamma(\alpha + \sigma)/\Gamma(\alpha). \tag{2.43}$$

For $\mu_1 = 0$, a similar analysis gives

$$\beta_r = \Gamma((r/\rho) - \theta - 1) \sum_{m=0}^{\leqslant r/\rho} \frac{\alpha_{r-m\rho} B_m}{m!\, \Gamma((r/\rho) - m - \theta)}. \tag{2.42}$$

III. Applications to Expansions in Orthogonal Polynomials

The same method can be used to sum series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) \tag{3.1}$$

when a_n can be represented by a Birkhoff series. This is because $P_n^{(\alpha,\beta)}(x)$ can be represented as a linear combination of two Birkhoff series,

$$P_n^{(\alpha,\beta)}(x) \sim n^{-1/2} \operatorname{Re} \left\{ Ce^{in\phi} \left[1 + \frac{\alpha_1}{n} + \frac{\alpha_2}{n^2} + \cdots \right] \right\} = \frac{V(n) + \overline{V(n)}}{2}, \tag{3.2}$$

$$x = \cos\phi, \ -\pi < \phi < \pi, \qquad n \to \infty.$$

The coefficients α_j can be calculated by applying the method of undetermined coefficients (described in the introduction) to the recurrence formula for $P_n^{(\alpha,\beta)}(x)$. The connecting constant C and $\bar{C}$, on the other hand, cannot be found this way, but can be read off the known asymptotic formula for $P_n^{(\alpha,\beta)}(x)$ [3, vol. 2, p. 198].

To terms of order n^{-2}, we have

$$C = \frac{e^{\gamma\phi i/2 - (\alpha/2 + 1/4)\pi i}}{\pi^{1/2} (\sin\phi/2)^{\alpha+2} (\cos\phi/2)^{\beta+2}}, \tag{3.3}$$

$$\alpha_1 = \frac{\left[\left(\frac{\gamma^2}{2} + \frac{1}{4} - \frac{\gamma}{2} \right) \cos\phi + \frac{(\alpha - \beta)(\gamma - 1)}{2} \right] - \left(\frac{\gamma}{2} + \alpha\beta \right) e^{-i\phi}}{-2i \sin\phi}, \tag{3.4}$$

$$\gamma = \alpha + \beta + 1.$$

We now write

$$a_n \sim e^{Q(\omega)} s(\omega), \tag{3.5}$$

and

$$\Delta R_n^{(1)} \sim V_1(n)\, e^{Q(\omega)} s(\omega), \qquad \omega \to \infty, \tag{3.6}$$

$$\Delta R_n^{(2)} \sim V_2(n)\, e^{Q(\omega)} s(\omega), \qquad \omega \to \infty. \tag{3.7}$$

Now V_1, V_2 can be converted into Birkhoff expansions in $\omega^{1/\rho}$ by letting $n = \omega - \xi$. The above equations then can be used to determine Birkhoff expansions for $R_n^{(1)}$ and $R_n^{(2)}$ since the product of two Birkhoff expansions of the above type is a Birkhoff expansion. Finally, we put

$$R_n = R_n^{(1)} + R_n^{(2)} \tag{3.8}$$

All the essential features of the procedure are displayed in the special case where $\alpha = \beta = -1/2$ (which corresponds to an expansion in Chebyshev polynomials) and $\rho = 1$, $Q(\omega) = \mu_1 \omega$.

We have

$$T_n(x) = (e^{in\phi} + e^{-in\phi})/2, \qquad x = \cos\phi, \tag{3.9}$$

i.e., the expansion (3.2) consists of a single term. Thus

$$\Delta R_n^{(1)} \sim \frac{-e^{i(\omega-\xi)\phi+\mu_1\omega}}{2}\, \omega^\theta \sum_{r=0}^{\infty} \alpha_r \omega^{-r}. \tag{3.10}$$

So

$$R_n^{(1)} \sim -e^{i\omega\phi+\mu_1\omega}\omega^\theta \sum_{r=0}^{\infty} \beta_r^{(1)} \omega^{-r} \tag{3.11}$$

where the $\beta_r^{(1)}$ satisfy the recursion relation (2.37) (with μ_1 replaced by $\mu_1 + i\phi$).

We have

$$\beta_0^{(1)} = \frac{-\alpha_0 e^{-i\xi\phi}}{2(e^{\mu_1+i\phi} - 1)}\ \beta_1^{(1)} = \frac{e^{-i\xi\phi}}{2}\left[\frac{-\alpha_1}{(e^{\mu_1+i\phi} - 1)} + \frac{\alpha_0\theta e^{\mu_1+i\phi}}{(e^{\mu_1+i\phi} - 1)^2}\right], \text{ etc.} \tag{3.12}$$

The coefficients in $R_n^{(2)}$ are found by replacing ϕ by $-\phi$ above.

$$\begin{aligned} &R_n \sim \omega^\theta e^{\mu_1\omega}[\alpha_0 v_{n,1} + (1/\omega)(\alpha_1 v_{n,1} + \theta\alpha_0 v_{n,2}) + \cdots], \quad n \to \infty, \quad \omega = n + \xi, \\ &v_{n,1} = \frac{T_n(x) - e^{\mu_1}T_{n-1}(x)}{X}, \\ &v_{n,2} = \frac{e^{\mu_1}(T_{n+1}(x) - 2e^{\mu_1}T_n(x) + e^{2\mu_1}T_{n-1}(x))}{X^2}, \\ &X = e^{2\mu_1} - 2xe^{\mu_1} + 1, \quad X \neq 0. \end{aligned} \tag{3.13}$$

As an illustration of this, consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\,\sigma^n T_n(x)}{n} = \operatorname{Re} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\,(\sigma e^{i\phi})^n}{n}$$

$$= \operatorname{Re} \ln(1 + \sigma e^{i\phi}) = \ln(1 + \sigma^2 + 2\sigma x)^{1/2}, \qquad |\,\sigma\,| < 1. \quad (3.14)$$

Here

$$\xi = 0,\ e^{\mu_1} = -\sigma,\ \theta = -1,\ \alpha_0 = 1,\ \alpha_j = 0, j > 0. \quad (3.15)$$

$$\ln(1 + \sigma^2 + 2\sigma x)^{1/2} = \sum_{r=1}^{n-1} \frac{(-1)^{r+1}\,\sigma^r T_r(x)}{r} + R_n \quad (3.16)$$

and

$$R_n \sim \frac{(-\sigma)^n}{n} \left\{ \frac{(T_n(x) + \sigma T_{n-1}(x))}{X} - \frac{\sigma}{n} \frac{(T_{n+1}(x) + 2\sigma T_n(x) + \sigma^2 T_{n-1}(x))}{X^2} + \cdots \right\}, \quad (3.17)$$

$$X = \sigma^2 + 2\sigma x + 1, \qquad n \to \infty.$$

Another interesting example is furnished by the series [3, vol. 2, p. 100].

$$J_\nu(2\sigma\,|\,x\,|) = \sum_{n=0}^{\infty} \epsilon_n J_{(\nu/2)-n}(\sigma)\, J_{(\nu/2)+n}(\sigma)\, T_{2n}(x), \quad \sigma > 0, \quad (3.18)$$

$$\epsilon_n = \begin{cases} 1, & n = 0, \\ 2, & n > 0. \end{cases}$$

If $\nu/2$ is integral, the series terminates. Otherwise, convergence is slow. Since

$$J_{(\nu/2)+n}(\sigma)\, J_{(\nu/2)-n}(\sigma) = \frac{(\sigma/2)^\nu\, \Gamma(n - \nu/2)(-1)^{n+1} \sin(\pi\nu/2)}{\Gamma((\nu/2) + n + 1)}$$

$$\times\ {}_2F_3 \left(\begin{matrix} 1/2 + \nu/1, \quad 1 + \nu/2 \\ 1 + n + (\nu/2), \quad 1 - n + (\nu/2), \quad 1 + \nu \end{matrix} \,\middle|\, -\sigma^2 \right) \quad (3.19)$$

a_n possesses a Birkhoff series representation in even powers of $1/n$ with $\mu_1 = \pi i$, $\theta = -\nu - 1$. The coefficients in the expansion can best be determined by using the work of Fields to find the asymptotic expansion of the ratio of gamma functions above and then expanding the individual terms of the ${}_2F_3$ in powers of $1/n^2$.

$$\epsilon_n J_{(\nu/2)+n}(\sigma)\, J_{(\nu/2)-n}(\sigma) \sim \frac{2(-1)^{n+1} \sin(\nu\pi/2)(\sigma/2)^\nu\, n^{-\nu-1}}{\pi}$$

$$\times \left[1 + \frac{1}{n^2} \left(\frac{\nu(\nu + 1)(\nu + 2)}{24} + \frac{\sigma^2(\nu + 2)}{4} \right) + \cdots \right], \quad n \to \infty. \quad (3.20)$$

Then

$$J_\nu(2\sigma \mid x \mid) = \sum_{r=1}^{n-1} \epsilon_r J_{(\nu/2)-r}(\sigma)\, J_{(\nu/2)+r}(\sigma)\, T_{2n}(x) + R_n\,, \tag{3.21}$$

$$R_n \sim \frac{2(-1)^{n+1}\sin(\nu\pi/2)(\sigma/2)^\nu n^{-\nu-1}}{\pi}\left\{\frac{(T_{2n}(x) + T_{2n-1}(x))}{2(1+x)}\right.$$
$$\left. - \frac{((\nu+1)\,T_{2n}(x) + 2T_{2n-1}(x) + T_{2n-2}(x))}{4(1+x)^2\, n} + \cdots\right\}, \quad n \to \infty, \quad x \neq -1. \tag{3.22}$$

This method of summing series of polynomials whose coefficients have Birkhoff series representations will work whenever the polynomials satisfy a difference equation with coefficients rational in n, since such polynomials themselves always have Birkhoff expansions. In particular, all the polynomials of hypergeometric type discussed by Fields and Luke [13, vol. 1, 7.4] and by Wimp [14] satisfy such difference equations. The method is particularly useful for summing series of Laguerre polynomials, since these expansions tend to converge rather slowly. Let $L_n^{(\alpha)}(x)$ be the Laguerre polynomial of degree n. Following the work of Fields and Luke [13, vol. 1, p. 264] we write

$$L_n^{(\alpha)}(x) \sim \operatorname{Re}\{A(x)\, e^{2i(nx)^{1/2}}\, n^{(\alpha/2)-1/4}\, [1 + C_1 n^{-1/2} + C_2 n^{-1} + C_3 n^{-3/2}]\}$$

$$A(x) = \frac{x^{-(\alpha/2)-1/4}}{\pi^{1/2}}\, e^{x/2} e^{-\pi i((\alpha/2)+1/4)}, \quad C_1 = i\psi_1 x^{-1/2}$$

$$C_2 = \frac{\psi_2}{x} - \frac{\psi_1^2}{2x} + \frac{\alpha(\alpha+1)}{2}, \tag{3.23}$$

$$\psi_1 = \frac{x^2}{12} - \frac{(\alpha+1)\,x}{2} - \frac{\alpha^2}{4} + \frac{1}{16},$$

$$\psi_2 = \frac{x^2}{16} - (\alpha+1)(2\alpha+1)\,\frac{x}{8} + \frac{\alpha^2}{16} - \frac{1}{64}, \quad x > 0, \quad \alpha > -1, \quad n \to \infty.$$

As a final example, consider the slowly convergent expansion for Tricomi's Ψ function [3, vol. II, p. 215]

$$\Gamma(a)\,\Psi(a, \alpha+1; x) = \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x)}{(n+a)}, \qquad a > 0, \qquad -1 < \alpha < 1/2. \tag{3.24}$$

Here we let

$$V_n \sim n^\theta e^{2i(nx)^{1/2}}\, [\beta_0 + \beta_1 n^{-1/2} + \beta_2 n^{-1} + \cdots], \tag{3.25}$$

$$\Delta V_n \sim -n^{(\alpha/2)-5/4} A(x)\, e^{i(nx)^{1/2}}$$
$$\times [1 + C_1 n^{-1/2} + (C_2 - a) n^{-1} + (C_3 - C_1 a) n^{-3/2} + \cdots], \tag{3.26}$$
$$R_n = \operatorname{Re} V_n . \tag{3.27}$$

Proceeding as previously, we find that

$$\Gamma(a)\, \Psi(a, \alpha + 1; x) = \sum_{k=0}^{n-1} \frac{L_n^{(\alpha)}(x)}{(n + a)} + R_n , \tag{3.28}$$

$$R_n \sim \frac{x^{-(\alpha/2)-3/4} e^{x/2} n^{(\alpha/2)-3/4}}{\sqrt{\pi}} \Big\{ \sin K_n(x) + (nx)^{-1/2} \Big[\Big(\frac{\alpha}{2} - \frac{3}{4} - \frac{x}{2}\Big) \cos K_n(x) + C_1 x^{1/2} \sin K_n(x) \Big] + (nx)^{-1} \Big[\Big(\Big(\frac{\alpha}{2} - \frac{5}{4}\Big) x^{1/2} - \frac{C_1}{2} x^{3/2} \Big) \cos K_n(x) - \Big(\frac{x^2}{12} - (C_2 - a)\, x + \Big(\frac{\alpha}{2} - \frac{3}{4}\Big)\Big(\frac{\alpha}{2} - \frac{5}{4}\Big) \Big) \sin K_n(x) \Big] + \cdots \Big\} ,$$
$$n \to \infty, \tag{3.29}$$

$$K_n(x) = 2(nx)^{1/2} - \pi \Big(\frac{\alpha}{2} + \frac{1}{4}\Big) , \; C_1 , C_2 \text{ as in (3.23).} \tag{3.30}$$

We could have taken $\omega = n + a$ rather than $\omega = n$, of course.

References

1. Jet Wimp and Y. L. Luke, Expansion formulas for generalized hypergeometric functions, *Rend. Circ. Mat. Palermo* **11** (1962), 351–366.
2. Y. L. Luke and Jet Wimp, Jacobi polynomial expansions of a generalized hypergeometric function over a semi-infinite ray, *Math. Comp.* **17** (1963), 395–404.
3. A. Erdélyi, (Ed.), "Higher Transcendental Functions," McGraw–Hill, N.Y., 1953.
4. Jet Wimp, The asymptotic representation of a class of G-functions for large parameter, *Math. Comp.* **21** (1967), 639–646.
5. Jet Wimp, On a certain class of Laguerre series, *Atti Ist. Ven. Sci.* **127** (1969), 141–148.
6. Jet Wimp, Recursion formulae for hypergeometric functions, *Math. Comp.* **22** (1968), 363–373.
7. J. L. Fields, Y. L. Luke, and Jet Wimp, Recursion formulae for generalized hypergeometric functions, *J. Approximation Theory* **1** (1968), 137–166.
8. C. R. Adams, On the irregular cases of the linear ordinary difference equation, *Trans. Amer. Math. Soc.* **30** (1928), 507–541.
9. P. M. Batchelder, "An Introduction to Linear Difference Equations," Dover, N.Y., 1967 (first published, 1927).
10. G. D. Birkhoff, Formal theory of irregular linear difference equations, *Acta Math.* **54** (1930), 205–246.

11. G. D. BIRKHOFF AND W. J. TRJITZINSKY, Analytic theory of singular difference equations *Acta Math.* **60** (1932), 1–89.
12. N. E. NÖRLUND, "Vorlesungen über die Differenzenrechnung, Chelsea, N.Y., 1954.
13. Y. L. LUKE, "The Special Functions and Their Approximations," Academic Press, N.Y., 1969.
14. JET WIMP, Recursion formulae for hypergeometric functions, *Math. Comp.* **22** (1968), 363–373.

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